Engineering Notes

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Equations of Motion of a Rotating Rigid Body

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I. Introduction

THE purpose of this Note is to comment on the equations of motion of a rigid body about a fixed point and their derivation in different forms. However, the central point is the nonorthogonality of the system of Euler angles and the implications of this fact. This nonorthogonality affects especially the linear relation between the angular velocity vector ω $(\omega_x,\omega_y,\omega_z)$ and the rates of change of the Euler angles $(\dot\varphi,\dot\theta,\dot\psi)$. It is well known that the matrix of this linear relation depends only on two of the Euler angles. We show a decomposition of this matrix in a product of two factors, each depending only on one of the two Euler angles. We comment on the meaning of this decomposition, in terms of the nonorthogonality of the system of Euler angles, and we show the consequences of this factorization. In particular, the quadratic form of the kinetic energy is decomposed in a product of seven factors. The corresponding Lagrangian equations of motion are then derived from it in tensor notations with the use of our computerized algebra program.

II. Notations and Definitions

In this section, we briefly summarize some of the basic equations in the theory of rotational motion, mostly to define our system of notations. We will use the three Euler angles (φ,θ,ψ), called, respectively, precession, nutation, and spin (or rotation) angles. These angles are used in the definition of the principal rotation matrix R, such that $r = Rr_I$, where r_I gives the position of a particle of the body relative to the fixed inertial axes and r is relative to the moving body axes. The rotation matrix is the product of three simple rotations, $R = R_\varphi R_\theta R_\varphi$ performed in the standard 3-1-3 order, following Goldstein's conventions.[1] We have, for instance,

$$R_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad R_\psi^T R_\psi = I \tag{1}$$

In what follows, we will frequently use the rates of change of the three Euler angles and, in order to abbreviate the writings, we define a column vector $\dot\varphi$ whose three components are $(\dot\varphi,\dot\theta,\dot\psi)$. Strictly speaking, this is not a vector but only a 3×1 matrix that helps to condense the equations. One of the principal results in the theory of rigid-body motion is the set of kinematic equations, which are the linear equations relating the rotation vector ω $(\omega_x,\omega_y,\omega_z)^T$ to the vector $\dot\varphi$. The rotation vector ω used here is defined relative to the body axes. Its inertial counterpart can also be defined and used, but it turns out that this is less useful. It is known that[1,2]

$$\omega = \begin{bmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \dot\varphi \\ \dot\theta \\ \dot\psi \end{bmatrix} = W\dot\varphi \tag{2}$$

Equation (2) clearly defines the W matrix. Note that this is not an orthogonal matrix, and we will come back to that later.

As for the other important equations of rotational motion, we remind the reader that R is orthogonal, that $RR^T = R^TR$ = identity, and that, on the basis of the time-derivative of R, we can define a skew-symmetric matrix $\Omega = \dot R R^T$. This matrix Ω contains only 0 and the three components ω_x, ω_y, and ω_z.

$$\Omega = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} \tag{3}$$

We also define the angular momentum $L = I\omega$, where I is the constant 3×3 diagonal matrix, with moments of inertia A, B, C. Euler's dynamical equations give the rate of change of the angular moment vector L as a function of the applied external torque N[3]:

$$\dot L = \Omega L + N \tag{4}$$

These equations are thus a direct consequence of the angular momentum equation in dynamics, $\dot L' = N'$, where the primes indicate a reference to the inertial axes. We mention, however, that other derivations of Eq. (4) are possible. For instance, they can be derived from Hamilton's principle, in the form of Euler-Lagrange equations, where the components $(\omega_x,\omega_y,\omega_z)$ are then considered as quasicoordinates.[3] We do not want to go into these details here. Equations (4), together with the kinematical equations (2), give the complete sixth-order system of equations of motion that needs to be integrated to determine the attitude of the rigid body as a function of time:

$$I\dot\omega = \Omega I\omega + N \tag{5a}$$

$$\dot\varphi = W^{-1}\omega \tag{5b}$$

Note again that all components of ω and N are here defined relative to the body axes. The torque vector N is usually a function of the Euler angles. The matrix W depends only on two Euler angles, ψ and θ, as was seen in Eq. (2).

III. Factorization of the W Matrix

It was said before that the W matrix, defined in Eq. (2), gives the linear transformation between the rates of the Euler angles and the angular velocity vector, $\omega = W\dot\varphi$. It was also said that this matrix is not orthogonal and depends on only two Euler angles. It is easy to verify that it can be factorized in a product of two matrices, R_ψ and S, each one containing only one Euler angle. Of the two factors, R_ψ is orthogonal and S is not. In other words, the lack of orthogonality is absorbed

Received March 22, 1987; revision received May 1, 1989.

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in a single simple matrix S, function of the nutation only:

$$W = R_\psi S, \qquad S = \begin{bmatrix} 0 & 1 & 0 \\ \sin\theta & 0 & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \tag{6}$$

The matrix S has the inverse

$$S^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ \sin\theta & 0 & 0 \\ 0 & -\cos\theta & \sin\theta \end{bmatrix} \Big/ \sin\theta \tag{7}$$

It is thus nonsingular, except when the nutation angle passes through 0 or π. It turns out that the preceding factorization is the basis for understanding many of the mysteries of rigid-body motion, in particular those stemming from the nonorthogonality of the Euler angle frame. One of the most striking illustrations is obtained by considering the length of the rotation vector ω:

$$\omega^2 = \omega^T\omega = \dot{\varphi}^T W^T W \dot{\varphi} = \dot{\varphi}^T S^T S \dot{\varphi} \tag{8}$$

which is a quadratic form in $(\dot{\varphi},\dot{\theta},\dot{\psi})$ with the symmetric matrix

$$S^T S = \begin{bmatrix} 1 & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \tag{9}$$

The length of the rotation vector ω is thus[2]:

$$\omega^2 = \dot{\varphi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\cos\theta \cdot \dot{\varphi}\dot{\psi} \tag{10}$$

This shows that, in terms of Euler angles, ω^2 is not a sum of squares or, in other words, that these angles do not form an orthogonal coordinate system (except again when the additional term vanishes, at $\theta = 0$ or π).

Another, and probably the most important, consequence and application of the preceding factorization is in the extreme simplicity of the two matrices R_ψ and S. This makes the derivation of many equations tractable, whereas the use of the more complex matrix W is more cumbersome. We will show this in the next section, in the derivation of the Lagrangian equations of motion, i.e., the three second derivatives $\ddot{\varphi},\ddot{\theta},\ddot{\psi}$, in tensor notations. However, to end the present section, we outline a direct derivation of these results in matrix notations.

Starting from Eq. (2) again, we see that

$$\ddot{\varphi} = (W^{-1})^{\cdot}\omega + W^{-1}\dot{\omega} = (W^{-1})^{\cdot}\, W\dot{\varphi} + W^{-1}I^{-1}(\Omega I\omega + N)$$

$$\ddot{\varphi} = (W^{-1})^{\cdot}\, W\dot{\varphi} + W^{-1}I^{-1}\Omega I W\dot{\varphi} + W^{-1}I^{-1}N \tag{11}$$

Formally speaking, this is the required result, although several lengthy simplifications are still required on the right-hand side in order to express all the remaining components of ω in terms of $\dot{\varphi}$. The approach presented in the following section is a more elegant solution, based on the factorization of W and the use of tensor notations.

IV. Second-Order Differential Equations for the Euler Angles

The second-order equations for the three Euler angles φ,θ,ψ are essentially the Euler-Lagrange equations derived from a Lagrangian $T + U$, where T is the kinetic energy of the rotating body and U the potential function. The kinetic energy T is known to be the half-dot product of the angular velocity ω and the angular momentum L:

$$L = I\omega = IW\dot{\varphi} = IR_\psi S\dot{\varphi} \tag{12}$$

$$T = ½\omega^T L = ½\omega^T I\omega = ½\dot{\varphi}^T S^T R_\psi^T I R_\psi S\dot{\varphi} \tag{13}$$

Not only do we have here an interesting factorization of the angular momentum in four factors but also a factorization of the kinetic energy in seven factors. All components are referred to principal body axes, and I is again the diagonal inertia matrix (A,B,C). The kinetic energy T is of course a quadratic form in $\dot{\varphi},\dot{\theta},\dot{\psi}$, with the symmetric matrix $S^T R_\psi^T I R_\psi S$ depending on two Euler angles. The derivation of the Lagrangian equations of motion simplifies if one notices that this quadratic form contains in it another simple quadratic form with the symmetric matrix $R_\psi^T I R_\psi$, depending only on the rotation angle ψ. We call this the rotational nucleus of the kinetic energy. It can be written as

$$R_\psi^T I R_\psi = \begin{bmatrix} a & d & 0 \\ d & b & 0 \\ 0 & 0 & c \end{bmatrix} \tag{14}$$

where

$$a = A\cos^2\psi + B\sin^2\psi \tag{15a}$$

$$b = A\sin^2\psi + B\cos^2\psi \tag{15b}$$

$$d = (A - B)\sin\psi\cos\psi \tag{15c}$$

Incidentally, we already see here the tremendous simplifications that occur in the case of symmetric bodies with $A = B$. In fact, the rotation angle disappears completely and the rotational nucleus reduces to a constant. We also mention that the determinant of the nucleus is ABC. The matrix is thus perfectly invertible. This inversion is necessary if one wants to use the Hamiltonian formulation, based on the contravariant metric tensor g^{ij}. We will not go into these details here.

Based on the preceding results, the complete quadratic form of the kinetic energy T can be written as follows (in tensor notations, this is also the twice-covariant form of the metric tensor g_{ij}):

$$g_{ij} = W^T I W = \begin{bmatrix} b\sin^2\theta + c\cos^2\theta & d\sin\theta & c\cos\theta \\ d\sin\theta & a & 0 \\ c\cos\theta & 0 & c \end{bmatrix} \tag{16}$$

The determinant is now $ABC\sin^2\theta$, and the matrix is invertible, except when the nutation angle passes through 0 and π. We also note that the precession angle φ is completely absent.

In order to write the equations of motion in tensor form, we will assume that the components 1, 2, 3 correspond, respectively, to the angles φ,θ,ψ (in this order!). The components of the metric tensor have been given earlier. The next step in the derivations is to construct the 27 first-kind Christoffel symbols Γ_{ijk}, which are simple functions of the partial derivatives of the nine g_{ij} components:

$$\Gamma_{ijk} = ½(g_{kj,i} + g_{ik,j} - g_{ij,k}) \tag{17}$$

The standard convention in tensor analysis is to indicate partial derivatives with respect to the ith coordinate with the symbol i. In the present problem, we find that 11 components are identically zero. The other 16 components are all given in Appendix A.

The second-kind Christoffel symbols are defined by

$$\Gamma_{ij}^k = g^{km}\Gamma_{ijm} \tag{18}$$

with an implied sum in the repeated index m. The explicit form of the equations of motion can then be given by the formula

$$\ddot{q}^i + \Gamma_{jk}^i \dot{q}^j \dot{q}^k = g^i v_{,j} \tag{19}$$

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where $\ddot{q}^i$ represents the coordinates φ, θ, ψ and where the standard summation convention applies. The expression on the left-side of the equal sign in Eq. (19) gives the contravariant components of the generalized acceleration.

In the case of the rigid-body motion expressed in Euler angles, five of the 27 Γ_{ij}^k symbols are identically zero, and the 22 others are given in Appendix B. Similar results were given in the simplified case $A = B$ by Luré.[5]

The different formulas and tensor components that have been given here were derived by hand as well as by our Poisson series manipulation system on the CDC-6600 computer.[4] To verify the algebra, the hand results were entered in the computer and subtracted symbolically from the computer results. These operations require the availability of three polynomial variables (A, B, C) and three harmonic variables (φ, θ, ψ). It is also necessary that the system be able to handle the negative exponents of the polynomial variables A, B, C, which occur frequently in the present application.

Appendix A: First-Kind Christoffel Symbols for the Rotating Rigid-Body Motion

$$\Gamma_{121} = \Gamma_{211} = \left[A \sin^2\psi + B \cos^2\psi - C\right] \sin\theta \cos\theta$$

$$\Gamma_{123} = \Gamma_{213} = -\tfrac{1}{2}\left[(A-B)(\cos^2\psi - \sin^2\psi) + C\right] \sin\theta$$

$$\Gamma_{131} = \Gamma_{311} = (A-B) \sin^2\theta \sin\psi \cos\psi$$

$$\Gamma_{132} = \Gamma_{312} = \tfrac{1}{2}\left[(A-B)(\cos^2\psi - \sin^2\psi) + C\right] \sin\theta$$

$$\Gamma_{231} = \Gamma_{321} = \tfrac{1}{2}\left[(A-B)(\cos^2\psi - \sin^2\psi) - C\right] \sin\theta$$

$$\Gamma_{232} = \Gamma_{322} = -(A-B) \sin\psi \cos\psi$$

$$\Gamma_{112} = -\left[A \sin^2\psi + B \cos^2\psi - C\right] \sin\theta \cos\theta$$

$$\Gamma_{113} = -(A-B) \sin^2\theta \sin\psi \cos\psi$$

$$\Gamma_{221} = (A-B) \cos\theta \sin\psi \cos\psi$$

$$\Gamma_{223} = (A-B) \sin\psi \cos\psi$$

Appendix B: Second-Kind Christoffel Symbols for the Rotating Rigid-Body Motion

$$\Gamma_{11}^1 = \frac{(A-B)(A+B-C)}{AB} \cos\theta \sin\psi \cos\psi$$

$$\Gamma_{12}^1 = \Gamma_{21}^1 = \frac{(A+B-C)}{2} \frac{\cos\theta}{\sin\theta} \left[\frac{\sin^2\psi}{A} + \frac{\cos^2\psi}{B}\right]$$

$$\Gamma_{13}^1 = \Gamma_{31}^1 = \frac{(A+B-C)(A-B)}{2AB} \sin\psi \cos\psi$$

$$\Gamma_{23}^1 = \Gamma_{32}^1 = \frac{1}{2AB \sin\theta} \left[A(A-B-C) - (A-B)(A+B-C) \sin^2\psi\right]$$

$$\Gamma_{11}^2 = \frac{-\sin\theta \cos\theta}{AB} \left[B(B-C) + (A-B)(A+B-C) \sin^2\psi\right]$$

$$\Gamma_{21}^2 = \Gamma_{12}^2 = \frac{(B-A)(A+B-C)}{2AB} \cos\theta \sin\psi \cos\psi$$

$$\Gamma_{31}^2 = \Gamma_{13}^2 = \frac{\sin\theta}{2AB} \left[B(A-B+C) - (A-B)(A+B-C) \sin^2\psi\right]$$

$$\Gamma_{32}^2 = \Gamma_{23}^2 = \frac{(B-A)(A+B-C)}{2AB} \sin\psi \cos\psi$$

$$\Gamma_{11}^3 = \frac{B-A}{ABC} \sin\psi \cos\psi \left[(A-C)(C-B) \cos^2\theta + AB\right]$$

$$\Gamma_{21}^3 = \Gamma_{12}^3 = \frac{-(A+B-C)\cos^2\theta}{2AB \sin\theta} \left[A + (B-A)\sin^2\psi\right] - \frac{\sin\theta}{2C} \left[(A-B)(\cos^2\psi - \sin^2\psi)\right] - \frac{\sin\theta}{2}$$

$$\Gamma_{22}^3 = \frac{A-B}{C} \sin\psi \cos\psi$$

$$\Gamma_{31}^3 = \Gamma_{13}^3 = \frac{-(A-B)(A+B-C)}{2AB} \cos\theta \sin\psi \cos\psi$$

$$\Gamma_{32}^3 = \Gamma_{23}^3 = \frac{-\cos\theta}{2AB \sin\theta} \left[-(A-B)(A+B-C)\sin^2\psi + A(A-B-C)\right]$$

$$\Gamma_{22}^1 = \Gamma_{33}^1 = \Gamma_{22}^2 = \Gamma_{33}^2 = \Gamma_{33}^3 = 0$$

References

[1]Goldstein, H., *Classical Mechanics*, Addison-Wesley, Reading, MA, 1957, p. 134.

[2]Desloge, E. A., *Classical Mechanics*, Wiley, New York, 1982, pp. 654, 658.

[3]Meirovitch, L., *Methods of Analytical Mechanics*, McGraw-Hill, New York, 1970, pp. 138, 157.

[4]Broucke, R., and Garthwaite, K., "A Programming System for Analytical Series Expansions on a Computer," *Celestial Mechanics*, Vol. 1, 1969, pp. 271-284.

[5]Luré, L., *Mécanique Analytique*, Masson, Paris, 1968, p. 640.

Navigation Path Planning for Autonomous Aircraft: Voronoi Diagram Approach

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Introduction

FOR autonomous aircraft and intelligent pilot aids, navigation path planning may be performed by computer rather than by humans. In this Note, we consider the task of planning a path from some start location to some finish location in mountainous terrain. The technique investigated employs searching for the best paths among topologically unique paths. Candidate paths are depicted by a search graph generated using a computerized geometric construct.

In reviewing approaches to navigation path planning, one must consider path-planning research involving robot manipulator arms, mobile robot and autonomous land vehicle path planning, and existing planning algorithms for pilot aid and autonomous aircraft.[1-6] For instance, the Dynapath algorithm[3] and Beaton et al.[4] generate path plans in fine detail us-

Received May 26, 1988; revision received Feb. 5, 1989.

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